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The Moyal-Dirac controversy revisited

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Abstract

In this paper we revisit the controversy that arose between Paul Dirac and Joe Moyal. Our motivation is provided by the fact that the algebraic aspects of both their approaches are becoming more appreciated as interest in the development of non-commutative geometry's attempt to "geometrise" quantum mechanics grows. Both were seeking to understand the role of non-commutativity, Moyal from a consideration of the differences between classical and quantum statistics, while Dirac was exploring its implications for the dynamics. The disagreement arose essentially over which should be given priority, the dynamics or the statistics. We will provide the background to show that both are essential aspects of the same overall mathematical structure.

1. Introduction

IN this paper I want to focus on the controversy that arose between Paul Dirac and Joe Moyal as a consequence of their different proposals for developing a phase space approach to quantum theory. This controversy gets to the heart of the fundamental problems in the description of quantum phenomena. Interestingly Dirac and Moyal were both "outsiders" in the sense that they started out as engineers, more so Moyal because of the way he entered academia, a story beautifully told in a biography penned by his wife Ann Moyal (2006).[1]

We pick up the story when Moyal was working in the Meteorological branch of the Ministère de l'Air in Paris at the outbreak of World War II. In the ensuing chaos, he managed to escape to England and was eventually given a job at the de Havilland Aircraft Company where he became the Assistant Director of the wartime Vibrations Department.

After the war Moyal wanted to work in quantum physics when fortunately an opportunity arose for him to enter academia by becoming an assistant lecturer in the Department of Mathematical Physics at Queen's University Belfast. It was there in 1949 that he published his classic paper which is the focus of his disagreement with Dirac.

This disagreement involves the question as how best to develop a (p, x, t) phase space formulation for quantum phenomena, the well-recognised difficulty arising from the non-commutativity of the quantum observables for momentum and position. In other words, if we replace the classical variables, (p, x), by operators, or rather elements of a non-commuting algebra, can we develop a coherent mathematical structure to describe quantum phenomena? Or must we simply use the wave theory as mathematically refined by the Hilbert space formalism with its interpretational difficulties?

Dirac showed how this phase space approach was possible by specifically developing his bra-ket notation for the purpose (Dirac 1939). In doing this, he

1. See also Moyal (2017). [Ed.]

showed that one had to introduce a special symbol, the standard ket, often overlooked, but necessary to distinguish algebraically specific representations in a Hilbert space. In the immensely fruitful Schrödinger approach one avoids any question about the standard kets by normalising the wave function at every stage. However, as Dirac (1965) himself points out, one can find situations where the Schrödinger picture fails because the state vector does not even remain in the same Hilbert space.

Moyal's approach was to use the structure of the algebra of functions on (p, x) phase space to compare a classical statistical theory with a quantum statistical theory. In doing so, he revealed the importance of a new bracket, the Moyal bracket, that replaced the quantum commutator bracket. Naturally, Moyal's (1949) paper focusses on *statistics* but he chose to discuss the relation between the algebraic approach, the wave function and the Schrödinger equation in an appendix at the end of the paper, giving the impression that statistics was more important than the dynamics. Dirac (1927, p. 641) on the other hand, did not think probability should be given priority over the dynamics. He writes:

> The notion of probabilities does not enter into the ultimate description of mechanical processes; only when one is given some information that involves a probability (e.g., that all points in η-space [η is a space of commuting variables] are equally probable for representing the system) can one deduce results that involve probabilities.

So already a misunderstanding began to arise fuelling a dispute. What neither of them had realised was that von Neumann (1931) had already created the algebra that Moyal developed in order to prove what became known as the Stone-von Neumann theorem, namely that the Schrödinger picture was unique, but only up to a unitary transformation. The theorem itself gave the impression that one need only work in this picture, so the wave function became *the way* of talking about non-relativistic quantum mechanics. The alternative Heisenberg picture, often called matrix mechanics, was thought only to be of importance for the relativistic domain and quantum field theory.

In this paper I want to bring out this historical background so that we see how the differences between Dirac and Moyal can be resolved. I will also show how other unitarily equivalent pictures arise and how they help to clarify a different overview of what both Dirac and Moyal were pioneering.

2. A Brief Historical Background to the Controversy

Mathematically, quantum theory had effectively two very different births. One in the mathematical work of Born and Jordan (1925) which was developed out of the physical insight of Heisenberg (1925) as he studied the emission of light quanta from accelerating electrons. This approach is generally known as matrix mechanics. The second approach emerged from Schrödinger's work using de Broglie's proposals that electrons should show the same interference effects as photons when they pass through two slits. This approach led to wave mechanics and the Schrödinger picture. Physically these two approaches looked very different. Schrödinger (1926) showed how they could be related mathematically, a relation that was later formalised in the Stone-von Neumann theorem. This mathematical fact still leaves open the question as to whether the Heisenberg and Schrödinger pictures are physically equivalent.

Wave mechanics used mathematical techniques that were very familiar to physicists at the time. Matrix mechanics, on the other hand, involved the new and unfamiliar mathematics of non-commutative structures — so much so that Heisenberg had to be told that he was using matrix multiplication.[2]

However, it was not that the general notion of non-commutativity was unfamiliar. For example, one must open the door before we can pass through; turn a book first through 90 degrees about the x-axis and then through 90 degrees about the y-axis and note its final orientation. Do the same thing in reverse order and you will obtain a different orientation. Again, measure the phase of a wave before measuring its amplitude and you will obtain a different result if you reverse the order of measurements. All very familiar. But what do we make of the non-commutability of position and momentum? The last example provides a simple answer, it is *all* to do with measurement. Hence *the* interpretation becomes *what we do*, rather than *what happens*.

Surely what happens naturally should not depend on what we do or don't do. The cosmos evolved before humans came into existence, so how do we understand the basic notion of movement, or of change? This was probably the most fundamental aspect of the disagreement between Dirac and Moyal, namely what was the best way to develop an ontology of quantum movement?

The classical ontology of movement asserted that a particle could be at position x and have momentum p simultaneously, so that a trajectory could be given a well-defined meaning. All this takes place in what is called a *phase space*. What happens when x and p become elements of a matrix algebra where they no longer commute and therefore cannot be given simultaneous meaning? Could they be given meaning without resorting to interpreting the symbols as describing our actions on the unfolding process?

Dirac (1945) certainly did believe that we could develop a theory that would provide us with a rather more definite picture of the motion of a quantum particle and indeed did make a specific proposal. The disagreement was not about developing a phase space approach to quantum phenomena but in the details of how we should construct such a theory. Indeed Dirac did make such a proposal but it turned out that the probabilities in his approach were, in general, complex numbers. In a comparison with the probabilities in the Moyal approach, Dirac (1945) concluded:

> Moyal's probability is always real, though not always positive, and is thus one step more physical than the probability of the present paper, but its region of applicability is rather restricted, as it does not seem to be connected with a general theory of functions like the present one.

Later in his classic textbook, Dirac (1947) made his overall position on probabilities in phase space very clear. After a discussion of the use of a probability density distribution, ρ, in a Gibbs ensemble in classical phase space he writes:

> We shall now see that there exists a corresponding density ρ in quantum mechanics having properties analogous to the above. It was first introduced by von Neumann. Its existence is rather surprising in view of the fact that phase space has no meaning in quantum mechanics, there being no possibility of assigning numerical values simultaneously to the q's and p's.

It should be noted that Moyal (1949) did not start from the dynamics, rather he focussed on the statistical aspects of the theory. His key question was "What are the similarities and differences between the statistical concepts used in quantum mechanics and those used in classical statistics?" He treated the time evolution of the

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2. By Max Born, in July 1925. See van der Waerden (1968), Introduction, p. 35. [Ed.]

statistics by starting from the Heisenberg equation of motion. As is made clear in the first quotation, it was the emphasis Moyal placed on the statistics, rather than the dynamics, that was the source of Dirac's objection.

As previously mentioned, both had overlooked a (1931) paper by von Neumann where he had developed the same non-commutative algebra that Moyal was exploring in his classic (1949) paper. Von Neumann's paper was the source of what became known as the Stone-von Neumann theorem which proves that the Schrödinger picture is unique up to a unitary transformation. It is remarkable that von Neumann used the algebra of the non-commutative phase space to reaffirm the Hilbert space structure he had set down in his classic (1932) book. The irony being that the Hilbert space formalism became so entrenched that those who tried to develop the algebraic approach, such as those using the $\bigstar$-algebra, were generally ignored. We will discuss the details of the von Neumann approach later in section 3.

2.1 The Mathematical Structure of the Moyal Approach

Let us start as Moyal (1949) did by comparing quantum statistics with the techniques used in classical statistics. In classical statistics, it is the *characteristic function* that plays a key role so Moyal set about constructing an analogous *quantum characteristic function* for a quantum system in a state ψ. This he did by first forming the operator

$$\widehat{M}(\tau,\theta) = \exp\,[i(\tau\hat{P} + \theta\hat{X})]. \tag{1}$$

Here $(\hat{P},\hat{X})$ are elements in the operator algebra satisfying the usual commutation relation

$$[\hat{X},\hat{P}] = i\hbar$$

and (τ,θ) are two commuting classical parameters (c-numbers).[3] Then the characteristic function in the state ψ is given by the scalar product

$$M_\psi(\tau,\theta) = \langle\psi\,|\,e^{i(\tau\hat{P}+\theta\hat{X})}\,|\,\psi\rangle. \tag{2}$$

Taking its Fourier inverse, we obtain the probability distribution function $F_\psi(p,x)$ so that

$$F_\psi(p,x) = \frac{1}{4\pi^2}\int\int \langle\psi\,|\,e^{i(\tau\hat{P}+\theta\hat{X})}\,|\,\psi\rangle\; e^{-i(\tau p+\theta x)} d\tau d\theta.$$

In this way Moyal arrived at the Wigner (1932) distribution function

$$F_\psi(p,x) = \frac{1}{2\pi}\int \psi^*(x-\tfrac{1}{2}\hbar\tau)\; e^{-i\tau p}\psi(x+\tfrac{1}{2}\hbar\tau)\,d\tau. \tag{3}$$

This shows that the variables (p,x) were actually the variables used in the Wigner distribution, so confirming that the variables are the elements in some non-commutative phase space.

What Moyal then shows is that the expectation value of any bounded operator, $\hat{A}$, can be simply found using the relation

$$\langle\hat{A}\rangle = \int\int \mathbf{a}(p,x)F_\psi(p,x,t)\,dx\,dp, \tag{4}$$

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3. The reduced Planck's constant, $\hbar$, is Planck's constant h divided by 2π. The angular momentum of any electron is an integral multiple of $\hbar$. [Ed.]

where $\mathbf{a}(p, x)$ is a function on the symplectic phase space. Moyal has implicitly assumed that the non-commuting operator algebra has been replaced by an algebra of C^∞-functions on a phase space. Thus the expectation value $\langle\psi_j|A|\psi_k\rangle$ can be obtained by integration of the ordinary function $\mathbf{a}(p, x)$ with respect to the corresponding phase space matrix $F_{jk}(p, x)$. Thus

$$\begin{aligned}\langle\psi_j\,|\,A\,|\,\psi_k\rangle &= \int\int A(p,x)F_{jk}(p,x)\,dp\,dx \\ &= \int\int\int\int \mathbf{a}(p,x)\,\langle\psi_j\,|\,e^{i(\tau\hat{P}+\theta\hat{X})}\,|\,\psi_k\rangle\,dp\,dx\,d\tau\,d\theta.\end{aligned}$$

The first surprise for Dirac (1947, p. 132) was that the (p, x) phase space appeared to be commutative and therefore the Heisenberg uncertainty principle would be violated. However, this turns out not to be the case provided we replace the commutator bracket, $i\hbar[\hat{R}\hat{G} - \hat{G}\hat{R}]$, by a new bracket, the Moyal bracket defined by

$$\frac{2}{\hbar}\sin\frac{\hbar}{2}\left[\frac{\partial}{\partial p_g}\frac{\partial}{\partial x_r} - \frac{\partial}{\partial p_r}\frac{\partial}{\partial x_g}\right]\mathbf{r}(p,x)\;\mathbf{g}(p,x) \tag{5}$$

where $\mathbf{r}(p, x)$ and $\mathbf{g}(p, x)$ are the phase space C^∞-functions that replace the two operators, $\hat{R}$ and $\hat{G}$. Notice even at this stage that the differential operator inside the square bracket has the same form as the classical Poisson bracket, a relation which we will develop further in section 3.3.

3. The von Neumann 1931 Paper

In the same year that his classic text appeared, von Neumann (1931) published the paper which formed the basis of the important Stone-von Neumann theorem. This paper is central to our discussion of the Moyal algebra. The importance of this theorem is that it proves that the Schrödinger picture is unique up to a unitary transformation. Thus it has provided the justification, quite rightly, for many physical situations, for the dominant use of the Schrödinger wave function picture, in spite of the well-known paradoxes and the unresolved "problem" of the collapse of the wave function.

It is interesting to take note, in passing, of von Neumann's confession to Birkhoff (Rédei 1996), saying that he no longer believed that the wave function should be regarded as an adequate description of the state of a quantum system. As a consequence Birkhoff and von Neumann (1936) developed a notion of what they called "Quantum Logic" to provide a different way of looking at quantum phenomena. As an algorithm, the Schrödinger picture has not been surpassed in the non-relativistic domain.

What is not generally realised is that the mathematical techniques that von Neumann used to prove his theorem are of major significance to Moyal's work. Indeed, what we will now show is that the mathematical structure developed by von Neumann is identical to the one that appeared in Moyal's classic paper (1949). In other words, the Moyal algebra is isomorphic to the standard operator algebra of quantum mechanics. This in turn implies that a *non-commutative phase space* can be regarded as lying at the heart of quantum theory.

Rather than starting with the well-known relation[4] $[\hat{X}, \hat{P}] = i$, von Neumann,

4. We will for convenience put $\hbar = 1$ in this section.

following Weyl (1927), introduces a pair of bounded operators, $U(\alpha) = e^{i\alpha\hat{P}}$ and $V(\beta) = e^{i\beta\hat{X}}$ so that the non-commutative multiplication can be written in the form

$$U(\alpha)V(\beta) = e^{i\alpha\beta}\, V(\beta)\, U(\alpha), \tag{6}$$

together with the relations,

$$U(\alpha)\, U(\beta) = U(\alpha+\beta); \quad V(\alpha)\, V(\beta) = V(\alpha+\beta).$$

One can now define an operator

$$\hat{S}(\alpha,\beta) = e^{-i\alpha\beta/2}\, U(\alpha)\, V(\beta) = e^{i\alpha\beta/2}\, V(\beta)\, U(\alpha)$$

which can also be written in the form

$$\hat{S}(\alpha,\beta) = e^{i(\alpha\hat{P}+\beta\hat{X})}. \tag{7}$$

This is exactly the operator $\widehat{M}(\tau,\theta)$ introduced by Moyal in his equation (1), provided we identify (τ,θ) with (α,β). Thus Moyal's mathematical starting point is exactly the same as that of von Neumann but is motivated from a very different standpoint.

Let us go further. Von Neumann then proves that the operator $\hat{S}(\alpha,\beta)$ can be used to define any bounded operator $\hat{A}$ on a Hilbert space through the relation

$$\hat{A} = \int\int \mathbf{a}(\alpha,\beta)\, \hat{S}(\alpha,\beta)\, d\alpha d\beta, \tag{8}$$

where $\mathbf{a}(\alpha,\beta)$ is the kernel of the operator.

To proceed further, von Neumann defines the expectation value of the operator $\hat{A}$ as

$$\langle\psi\,|\,\hat{A}\,|\,\psi\rangle = \int\int \mathbf{a}(\alpha,\beta)\, \langle\psi\,|\,\hat{S}(\alpha,\beta)\,|\,\psi\rangle\; d\alpha\, d\beta. \tag{9}$$

Here

$$\langle\psi\,|\,\hat{S}(\alpha,\beta)\,|\,\psi\rangle = \langle\psi\,|\,e^{i(\alpha\hat{P}+\beta\hat{X})}\,|\,\psi\rangle$$

so that

$$\langle\psi\,|\,\hat{S}(\alpha,\beta)\,|\,\psi\rangle = \langle\psi\,|\,\widehat{M}(\alpha=\tau,\beta=\theta)\,|\,\psi\rangle,$$

which, apart from a change of variables, is identical to the expression used by Moyal in equation (2). If we now use the Fourier transformation of $M_\psi(\tau,\theta)$ in equation (9), we find it immediately gives the Moyal equation (4) for the expectation value. Including the time dependence, the expectation values of the two approaches give

$$\langle\psi\,|\,\hat{A}(t)\,|\,\psi\rangle = \int\int \mathbf{a}(p,x,t)F_\psi(p,x,t)\, dp\, dx. \tag{10}$$

Here $\mathbf{a}(p,x,t)$ is called the *Weyl symbol* (de Gosson 2016), its Fourier transform being $\mathbf{a}(\tau,\theta,t)$.

Thus Moyal has chosen to label the parameters in the Fourier inversion (p,x) because he is anticipating a generalised phase space. Von Neumann, on the other hand, attaches no specific physical meaning to the parameters (α,β).

3.1 The Relationship Between Quantum Operators and Weyl Symbols

Looking at equation (8) we see that there is a well-defined relationship between an operator, $\hat{A}$, and its corresponding symbol, $\mathbf{a}(\alpha,\beta)$. If the algebraic structure of the quantum operators is to be made isomorphic to the algebraic structure inherited by the symbols, we must find the nature of the two defining binary relations between the symbols.

Clearly addition, being abelian, is straightforward so that

$$\hat{A} + \hat{B} \to \mathbf{a}(\alpha, \beta) + \mathbf{b}(\alpha, \beta)\ .$$

The product, being non-commutative, is more difficult and we must find how the product $\hat{A}\hat{B} = \hat{C}$ translates into the product $\mathbf{a}(\alpha, \beta) \odot \mathbf{b}(\alpha, \beta) = \mathbf{c}(\alpha, \beta)$ so that the expectation value for the product is the same in both cases. To show how this is possible, we follow von Neumann and write

$$\begin{aligned}
\langle g \,|\, \hat{A}\hat{B} \,|\, f \rangle = \langle \hat{A}^* g \,|\, \hat{B} f \rangle &= \int\!\!\int \mathbf{b}(\alpha, \beta)\, \langle \hat{A}^* g \,|\, \hat{S}(\alpha, \beta) f \rangle\; d\alpha d\beta \\
&= \int\!\!\int \mathbf{b}(\alpha, \beta)\, \langle g \,|\, \hat{A}\hat{S}(\alpha, \beta) f \rangle\; d\alpha d\beta \\
&= \int\!\!\int\!\!\int\!\!\int \mathbf{b}(\alpha, \beta)\, e^{\frac{1}{2} i(\gamma\beta = \delta\alpha)}\, \mathbf{a}(\gamma - \alpha, \delta - \beta)\, \langle g \,|\, \hat{S}(\gamma, \delta) f \rangle\; d\alpha d\beta d\gamma d\delta \\
&= \int\!\!\int \left[\int\!\!\int e^{\frac{1}{2} i(\gamma\beta - \delta\alpha)}\, \mathbf{a}(\gamma - \alpha, \delta - \beta)\, \mathbf{b}(\alpha, \beta)\, d\alpha\, d\beta \right] \langle g \,|\, \hat{S}(\gamma, \delta) f \rangle\;\; d\gamma\, d\beta.
\end{aligned}$$

The kernel of $\hat{A}\hat{B}$ is thus $\int\!\!\int e^{\frac{1}{2} i(\gamma\beta - \delta\alpha)} \mathbf{a}(\gamma - \alpha, \delta - \beta)\, \mathbf{b}(\alpha, \beta) d\alpha d\beta$. (The absolute integrability of this expression follows from the deduction.) Von Neumann's product can be transformed into one given in terms of the variables (p, x). We thereby arrive at the ★-product which can be written in a geometrically illuminating form (Hirshfeld and Henselder 2002a)

$$\begin{aligned}
&\mathbf{a}(p, x) \bigstar\ \mathbf{b}(p, x) \\
&\quad = (\pi\hbar)^{-2} \int\!\!\int\!\!\int\!\!\int \exp\left[\left(\frac{2}{i}\, \hbar \right) (p(x_1 - x_{2)} + x(p_2 - p_1) + (x_2 p_1 - x_1 p_2)) \right] \qquad (11) \\
&\quad\quad \times \mathbf{a}(p_1, x_1)\, \mathbf{b}(p_2, x_2)\, dp_1 dp_2 dx_1 dx_2.
\end{aligned}$$

The exponent can then be simplified by writing $z = (p, x)$ to give

$$\frac{1}{2}\, [p(x_2 - x_1) + x(p_1 - p_{2)} + (x_1 p_2 - x_2 p_1)] = \frac{1}{2}\, (z - z_1) \wedge (z - z_2) = A(z, z_1, z_2)$$

where $A(z, z_1, z_2)$ is an area, a symplectic area in phase space (see Figure 1).

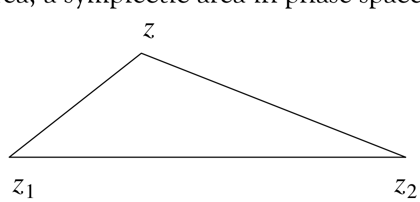

Figure 1: Symplectic area $A(z, z_1, z_2)$

We then find that the ★-product can be written in the form

$$(\mathbf{a} \bigstar \mathbf{b})(z) = \int\!\!\int \exp\left[\frac{4i}{\hbar}\, A(z, z_1, z_2) \right] \mathbf{a}(z_1)\, \mathbf{b}(z_2)\, dz_1 dz_2.$$

Thus the ★-product is non-local in that it involves integrating over a non-local region in the non-commutative phase space. It is this product that is used in M-theory. Notice that when the area $A(z, z_1, z_2)$ is zero, the multiplication is commutative and we return to the classical domain. For an excellent and more extensive discussion of the ★-product and the implications of its non-local nature see Zachos (2000, and 2002) and Zachos, Fairlie and Curtright (2005, and 2014).

3.2 The Non-local ★-Product

In an analysis that focusses on the *non-local* nature of the ★-product, Hiley (2015) shows that the (p, x) should be identified with the mean position of a "blob" in phase space (de Gosson 2013). To motivate this suggestion we follow the work of Berezin and Shubin (2012) who show that there is a relation between propagators in space-time and phase space kernels. A similar result was proposed by Bohm and Hiley (1981, and 1983), who worked from a different perspective.

If $K(y, y')$ is the propagator linking two points, (y, y'), in configuration space and $F(p, x)$ is the corresponding phase space kernel, which Moyal calls the "phase space distribution," we have the relations

$$K(y, y') = \int\int L^*(y, y' \mid p, q)F(p, q)dpdq$$

and

$$F(p, x) = \int\int L(p, x \mid y, y')K(y, y')\,dyd y'.$$

After some detailed work that can be found in Berezin and Shubin (2012), we find the function $L(p, x|y, y')$ and obtain the relations in their final form

$$K(x, y) = (2\pi\hbar)^{-n}\int F(p, x)e^{-ip.(y-x)/\hbar}dp$$

and

$$F(p, x) = \int K(x, y)e^{ip.(y-x)/\hbar}dy.$$

Rather than following von Neumann to arrive at (11), we can obtain the formula for the product of kernels by considering the succession of propagators which form a groupoid defined by

$$K(y, y') = \int K_1(y, z)K_2(z, y')\,dz.$$

In this way we arrive at an expression for the product of kernels (11), again confirming the isomorphism between the operator algebra and the non-commutative algebra formed by functions on a quantum phase space.

Remember that (y, y') are the coordinates of two separate points in configuration space which means we are also considering two points in phase space, (p, y) and (p', y'). In $2n$-dimensional phase space we have what de Gosson (22) calls a "blob." The main measure of such a blob is its symplectic capacity or phase space area (in two-dimensional phase space). These are the type of object that lie at the heart of M-theory (Steinacker 2011).

Now we make the coordinate transformations

$$p \to P = (p + p')/2 \;\text{ and }\; x \to X = (y + y')/2$$

while

$$\tau = y - y' \;\text{ and }\; \theta = p - p'.$$

The important conclusion we then arrive at is that the Moyal algebra is describing extended objects in phase space. In other words, the quantum formalism translates to something *non-local* on a phase space. This is a key point that will come up again and again. For now we will simply treat it as a mathematical consequence of the Moyal formalism. However, it should already be noted that Dirac's criticism was based on the implicit assumption of a local description in phase space. This then surely resolves one difficulty that Dirac anticipated because he assumed a local phase space description whereas we are concerned with "areas" or "regions" of phase space.

3.3 More on the ★-Product

The fact that Moyal can use C^{∞}-functions to describe quantum phenomena will generate, as our own experience shows, disbelief so we feel it is necessary to go into a few more details concerning the ★-product and the implications of the formalism. This radical change in the algebraic structure becomes more compelling once we realise that the ★-product can be written as

$$\mathbf{a}(p,x)\star\mathbf{b}(p,x)=\mathbf{a}(p,x)\exp\left[\frac{i\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial p}-\frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial x}\right)\right]\mathbf{b}(p,x) \tag{12}$$

which is just the complex exponential of the classical Poisson bracket (Groenewold 1946). Quantum mechanics is not "another world," as the "classical world" actually emerges from the underlying quantum processes. Thus while classical mechanics involves the representations of the symplectic and orthogonal groups, quantum mechanics exploits the representations of their covering groups, explaining the appearance of the orthogonal and symplectic spinors.

Moyal only considered the symplectic aspects of the symmetries in his investigations, so his results were pertinent to the double cover of the symplectic group, namely, the metaplectic group and its non-linear generalisation (Guillemin and Sternberg 1984).

It was Groenewold (1946) who first wrote down equation (12) in terms of its trigonometric expansion. He writes

$$\mathbf{a}(p,x)\,\frac{2}{\hbar}\sin\frac{\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial p}-\frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial x}\right)\mathbf{b}(p,x)\leftrightarrow\frac{i(\mathbf{ab}-\mathbf{ba})}{2}$$

which was used by Moyal in the form of equation (5). The remaining cosine term

$$\mathbf{a}(p,x)\,\frac{2}{\hbar}\cos\frac{\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial p}-\frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial x}\right)\mathbf{b}(p,x)\leftrightarrow\frac{\mathbf{ab}+\mathbf{ba}}{2} \tag{13}$$

was not used by Moyal.

An exponential form is extremely useful for cases where the $\mathbf{a}(p,x)$ and $\mathbf{b}(p,x)$ are finite polynomials. For example, it is trivial to show that

$$x\star p-p\star x=i\hbar.$$

This demonstrates that a form of the Heisenberg commutator also appears in the algebra as it must. It is often convenient to write the ★-product in terms of two types of bracket. The first is the Moyal bracket defined by

$$\{\mathbf{a}, \mathbf{b}\}_{MB} = \frac{\mathbf{a} \star \mathbf{b} - \mathbf{b} \star \mathbf{a}}{i} \hbar. \tag{14}$$

The second is a Jordan product, which we have elsewhere called the Baker bracket (Baker 1958) for historical reasons. It is defined by

$$\{\mathbf{a}, \mathbf{b}\}_{BB} = \frac{\mathbf{a} \star \mathbf{b} + \mathbf{b} \star \mathbf{a}}{2}. \tag{15}$$

A series expansion of the $\star$-product will produce a power series in $\hbar$ which forms the basis for *deformation quantum mechanics* (Hirshfeld and Henselder 2002a). A more mathematically advanced treatment will be found in Khalkhali (2009). If we retain only the terms to $O(\hbar)$, we find

$$\text{Moyal bracket} \rightarrow \text{Poisson bracket to } O(\hbar)\,.$$

This bracket is defined in equation (7.8) of Moyal's (1949) paper. Thus classical mechanics emerges from this structure if we only retain terms to $O(\hbar)$. While in the case of the Baker bracket, we find

$$\text{Baker bracket} \rightarrow \text{commutative bracket to } O(\hbar)\,.$$

Hence it is only when going to $O(\hbar^2)$ and above that quantum effects emerge from the Jordan product.

Moyal makes no use of the Baker bracket, but Baker (1958) shows that for a pure state, the Green's function is degenerate and can be written in the form $K(y, y') = g^*(y)g(y')$. Thus the wave function appears only when the propagator is degenerate. It was from this form that Baker showed that we could write

$$F(p, x) = \hbar \left(\frac{i}{2} \, [F, F]_{MB} + [F, F]_{BB} \right)$$

where $[F, F]_{MB}$ is the Moyal bracket and $[F, F]_{BB}$ is the Baker bracket.

Clearly the Moyal bracket replaces the quantum operator commutation relations $[\hat{A}, \hat{B}]$. It is this bracket that was used by Moyal in deriving the continuity equation which we will use in section 5.1. On the other hand if the expansion of the Baker bracket is limited to $O(\hbar)$ then it reduces to the usual commutative product. It was for this reason that Dirac (1947) missed the appearance of the quantum potential energy. Whereas it appears in the appendix of Moyal's classic (1949) paper, as we will show in section 5.1.

To repeat, it is only when we go to order $O(\hbar^2)$ and above that the Baker bracket does not reduce to the usual commutative product. Generally terms of $O(\hbar^2)$ are assumed to be negligible and therefore are not discussed, but the bracket plays an important role when energy (Hiley 2015) is involved. A careful study of Pauli's (1926) application of the algebraic approach to the energy level structure of the hydrogen atom shows how a Jordan product enters into the calculation.

As we have already pointed out, one of the advantages of the Moyal approach is that it contains classical physics as a limiting case as is clearly seen from equation (12). There is no need to look for a one-to-one correspondence between commutator brackets and Poisson brackets, a process which fails as was demonstrated by the well-known Groenewold-van Hove "no-go" theorem (Guillemin and Sternberg 1984). Furthermore, it is not necessary to introduce the notion of decoherence as a fundamental process in order to obtain the classical limit. This does not mean that decoherence has no role to play in quantum physics. It plays a vital role in real experiments where noise and other thermal processes enter to destroy quantum interference. However, destroying the interference does not necessarily return us to the classical formalism involving Poisson brackets. It merely destroys coherence.

3.4 The Physical Meaning of the Weyl Symbol

To complete this section, let us examine the physical meaning of the Weyl symbol $\mathbf{a}(p,x,t)$ introduced in equation (10) in more detail. We start with the standard definition of the mean value of the operator $\hat{A}$,

$$\langle \hat{A}\rangle = \langle \psi(t)\,|\,\hat{A}\,|\,\psi(t)\rangle = \int\int \langle \psi(t)\,|\,x'\rangle\;\langle x'\,|\,\hat{A}\,|\,x''\rangle\;\langle ix''\,|\,\psi(t)\rangle\;dx'dx''.$$
$$= \int\int \langle x'\,|\,\hat{A}\,|\,x''\rangle\;\rho\,(x',x'',t)\,dx'dx''.$$

Let us now change coordinates using $x' = x - \tau/2$ and $x'' = x + \tau/2$, then

$$\langle \hat{A}\rangle = \int\int \langle x-\tau/2\,|\,\hat{A}\,|\,x+\tau/2\rangle\;\rho(x-\tau/2, x+\tau/2, t)\,dx\,d\tau.$$

Now write $\rho\,(x-\tau/2, x+\tau/2, t) = \int F_\psi(p,x,t)e^{ip\tau}dp$, and we find

$$\langle \hat{A}\rangle = \int\int\int \langle x-\tau/2\,|\,\hat{A}\,|\,x+\tau/2\rangle\;e^{-ip\tau}d\tau[F_\psi(p,x,t)\,dp\,dx]$$

which becomes equation (10) if we identify

$$\mathbf{a}(p,x,t) = \int \langle x-\tau/2\,|\,\hat{A}(t)\,|\,x+\tau/2\rangle\;e^{-ip\tau}d\tau.$$

Thus we see that $\mathbf{a}(p,x,t)$ is derived from a transition probability amplitude integrated over the "blob" at the mean position x when the blob is moving with mean momentum p. The Weyl symbol $\mathbf{a}(p,x,t)$ is sometimes called the "classical observable" associated with the observable $\hat{A}$, but I find that association misleading since there is very little that is classical about $\mathbf{a}(p,x,t)$.

Moyal noticed that if $\mathbf{a}(p,x,t)$ could be regarded as one of the possible values of $\hat{A}$ and if we could regard $F_\psi(p,x,t)$ as a probability distribution, then the RHS of (10) has exactly the form of a classical expectation value where $F_\psi(p,x,t)$ is a weighting function. So why not treat $F_\psi(p,x,t)$ as a probability distribution? After all, we can write equation (3) in a slightly different form

$$F_\psi(p,x,t) = \frac{1}{2\pi}\int e^{-ip\tau}\langle x-\tau/2,t\,|\,(\,|\,\psi\rangle\;\langle\psi\,|\,)\,|\,x+\tau/2,t)\,d\tau.$$

Notice $(|\psi\rangle\;\langle\psi|)$ is just the density operator, $\hat{\rho}$, for a pure state, so that we can write

$$\rho_\psi(p,x,t) = \frac{1}{2\pi}\int e^{-ip\tau}\;\langle x-\tau/2,t\,|\,\hat{\rho}\,|\,x+\tau/2,t\rangle\;d\tau. \tag{16}$$

Thus clearly demonstrating that the probability distribution $F_\psi(p,x,t)[:=\rho_\psi(p,x,t)]$ is simply the Weyl symbol of the density matrix for a pure state in the (p,x,t) representation.

This has then been used to argue against the whole approach because $F_\psi(p,x,t)$ will always be negative somewhere in phase space when quantum effects show up. Moreover this has generated much debate with Bartlett (1945) and even Feynman (1987) feeling it necessary to defend the use of Wigner functions which may be negative. But we should not even be having the argument because $\mathbf{a}\,(p,x,t)$ is not a "classical observable." It is an average over a region in phase space. Note there is no reason why a density matrix (16) should stay positive. The positivity condition is only desirable if $F_\psi(p, x, t)$ is to be regarded as a probability density.

4 The Bohm approach

The Bohm (1952) approach has a deep connection with Moyal's work. This is highlighted by the fact that the two key equations of Bohm's theory already appear in the appendix of Moyal's (1949) paper. Moreover, Dirac's classic (1947) book also contains a harbinger of the Bohm approach.[5] Dirac obtains the quantum Liouville equation but does not exploit the quantum Hamilton-Jacobi equation, the real part of the Schrödinger equation. Subsequently Dirac only explored his version of the algebraic approach by expanding the formalism to $O(\hbar)$ and so only recovered the classical Hamilton-Jacobi equation. Why he did not explore terms of $O(\hbar^2)$ when the quantum Hamilton-Jacobi [QHJ] equation appears is not clear. He simply writes (1947):

> By a more accurate solution of the wave equation one can show that the accuracy with which the coordinates and momenta simultaneously have numerical values cannot remain permanently as favourable as the limit allowed by Heisenberg's principle of uncertainty $\cdots$, but if it is initially so it will become less favourable, the wave packet undergoing a spreading.

Later Bohm showed that, by using the Schrödinger picture, there was no conflict with the uncertainty principle.

Because of the title Bohm chose for his (1952) paper, his work became entangled in the "hidden variable" controversy, which is unfortunate, as no new variables were added to the standard formalism and this old controversy has deflected attention away from the real implications of the physics lying behind the Bohm method.

As far as the mathematical structure is concerned, the only novelty Bohm introduces is in the *interpretation* of the mathematical symbols used in the Schrödinger picture. Mathematically, the approach simply uses the Schrödinger equation and separates it into its real and imaginary parts under polar decomposition of the wave function.[6]

Bohm's paper focusses attention on the ideas already presented in Schrödinger's (1952) paper "Are there quantum jumps?" Schrödinger argues that a description in terms of a continuous evolution should be possible "without losing either the precious results of Planck and Einstein on the equilibrium of (macroscopic) energy between radiation and matter, or any other understanding of phenomena that the parcel-theory [sic quanta] affords."

For the purposes of this paper we need only know that the real part of the Schrödinger equation under the polar decomposition of the wave function can be written in the form

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5. The specific section in Dirac (1947) we are referring to here is §21, entitled "The motion of wave packets." A more detailed discussion of this relationship will be found in Hiley and Dennis (2018, 2019).

6. A first draft of Bohm's paper has recently come to light in the Archive Louis de Briglie at the French Academy of Science. The original title of the paper was "A Causal and Continuous Interpretation of the Quantum Theory," a title which more accuractely reflects the conetnt of the paper (Drezet and Stock 2021).

$$\frac{\partial S(x,t)}{\partial t} + \frac{1}{2m}(\nabla S(x,t) - e\mathbf{A})^2 + \hbar^2 Q_\psi(x,t) + V(x,t) = 0 \tag{17}$$

where $Q_\psi(x,t) = -\nabla^2 R(x,t)/2mR(x,t)$. Bohm (and also de Broglie (1960)) called this term the "quantum potential energy." This new quality of energy enters as the coefficient of $\hbar^2$ and this is why Dirac missed the QHJ equation. Its appearance is intimately connected with the Baker bracket (Jordan product) and therefore the non-commutativity of $(x \bigstar p)$.

To understand the meaning of this equation, recall that in classical physics the canonical energy is given by $E = -\frac{\partial S}{\partial t}$ while the canonical momentum is given by $p = \nabla S$, so that equation (17) can be regarded as the quantum equivalent of an energy conservation equation. This means that in the quantum domain a new quality of energy appears, namely, the quantum potential energy.

We should not be surprised that a new quality of energy is involved because the quantum vacuum is a sea of virtual particle-antiparticle pairs. At higher energies these virtual particles emerge as real particle-antiparticle pairs. In this case we are in exactly the same situation that chemists find themselves in when having to deal with a many-particle system. Here thermodynamics with its various qualities of energy such as Helmholtz free energy, Gibbs free energy and even heat energy have to be distinguished and accounted for. In this context it seems eminently sensible to take the possibility of a new quality of energy seriously in the quantum domain.

5 Back to Moyal

It may seem that we have strayed from the Moyal-Dirac disagreement over a phase space description by bringing in a discussion of the Bohm model. However, the main equations that Bohm used appear already in the appendices A1 and A2 of Moyal's 1949 paper. This naturally raises the question as to the nature of the relation between the two approaches.

In appendix A1, Moyal (1949) introduces the space-conditional average of the momentum and obtains the relation, $\overline{\overline{p}} = \nabla S$ where S is the phase of the wave function. This is identical to the momentum, $p_B = \nabla S$, introduced in the de Broglie-Bohm approach.

To show this we use the distribution function, $F_\psi(p,x,t)$, to construct the space-conditional moments of the momentum. These are written as $\overline{\overline{p^n}}$ and defined through the general formula

$$\rho(x)\,\overline{\overline{p^n}}(x) = \int p^n F_\psi(p,x,t)\,dp. \tag{18}$$

This can then be written in the form

$$\rho(x)\,\overline{\overline{p^n}}(x) = \left(\frac{\hbar}{2i}\right)^n \left[\left(\frac{\partial}{\partial x^1} - \frac{\partial}{\partial x_2}\right)^n \psi(x_1)\,\psi^*(x_2)\right]_{x_1=x_2=x} \tag{19}$$

where $\rho(x) = \int F_\psi(p,x,t)\,dp = \psi^*(x)\,\psi(x)$. If we now write

$$\psi(x) = \rho^{1/2}(x)\,e^{iS(x)/\hbar} \tag{20}$$

we find for $n = 1$

$$\overline{\overline{p}} = \nabla S. \tag{21}$$

This is equation (A 1.6) in the appendix of the Moyal 1949 paper. Note that this means $\overline{\overline{p}}$ is dependent on the wave function ψ as is the de Broglie-Bohm momentum. Please also note we are not interpreting this as a "guidance condition"; it was de Broglie who later regretted introducing such a concept.

5.1 The Transport of $\overline{\overline{p}}$

Now we are in a position to show how Moyal described the dynamics. To obtain a transport equation for $\overline{\overline{p}}$, we need the equation for the time development of the quasi-probability distribution. This is written in the form

$$\frac{\partial F(p,x,t)}{\partial t} + \{F(p,x,t), H(p,x)\}_{MB} = 0 \tag{22}$$

where $\{F(p,x,t), H(p,x)\}_{MB}$ is the Moyal bracket defined in equation (14). We have omitted the subscript ψ on $F(p,x,t)$ because this equation is valid for all wave functions. In the limit to $O(\hbar)$, equation (22) becomes the classical Liouville equation.

With a specific Hamiltonian, $H(p,x) = p^2/2m + V(x)$, we can show that equation (22) leads to the real part of the Schrödinger equation (17) used in the Bohm approach. To do this we write equation (22) in the form

$$\frac{\partial F(p,x,t)}{\partial t} + \frac{p}{m}.\nabla F(p,x,t) = \int J(p-p',x)F((p',x,t)\,dp' \tag{23}$$

where

$$J(p-p',x) = -i\int[V(x-y/2) - V(x+y/2)]e^{i(p-p')y}dy. \tag{24}$$

The full details of the derivation of this result can be found in Takabayasi (1954).

To obtain the expression for the transport equation for $\overline{\overline{p}}$, we must multiply equation (23) by p_k to obtain

$$\frac{\partial p_k F(x,p,t)}{\partial t} + \sum \frac{p_k p_i}{m}\frac{\partial F(x,p,t)}{\partial x_i} = \int p_k J(x,p-p')F(x,p',t)dp'. \tag{25}$$

By introducing a wave function, $\psi(x,t) = R(x,t)\exp(iS(x,t)/\hbar)$, and then integrating over p, we find the RHS of equation (23) reduces to $-\rho\,\partial V/\partial x_k$ where $\rho = R^2$. Then equation (25) becomes

$$\frac{\partial(\rho\,\overline{\overline{p^k}})}{\partial t} + \frac{1}{m}\sum_i \frac{\partial}{\partial x_i}(\rho\,\overline{\overline{p_i p_k}}) = -\rho\frac{\partial V}{\partial x_k}. \tag{26}$$

We can also show that the dispersion in momentum becomes

$$\frac{1}{m}\sum_i \frac{\partial}{\partial x_i}[(\rho\,\overline{\overline{p_i p_k}}) - (\rho\,\overline{\overline{p_i}}.\overline{\overline{p_k}})] = -\frac{\hbar^2}{4m}\sum_i\frac{\partial}{\partial x_i}\left[\rho\frac{\partial^2 \ln\rho}{\partial x_i \partial x_k}\right]. \tag{27}$$

This result may now be used in equation (26) so that it can be written in the form

$$\frac{\partial(\rho\,\overline{\overline{p_k}})}{\partial t} + \frac{1}{m}\sum_i \frac{\partial(\rho\,\overline{\overline{p_i}}.\overline{\overline{p_k}})}{\partial x_k} = -\rho\frac{\partial V}{\partial x_k} - \frac{\hbar^2}{4m}\sum_i\frac{\partial}{\partial x_i}\left[\rho\frac{\partial^2\ln\rho}{\partial x_i\partial x_k}\right]. \tag{28}$$

Differentiating the first term in equation (28) and using equation (21) we find

$$\rho \frac{\partial}{\partial x_k}\left[\frac{\partial S}{\partial t} + \frac{1}{2m}\sum_i \left(\frac{\partial S}{\partial x_i}\right)^2 + V\right] = \frac{\hbar^2}{4m}\sum_i \frac{\partial}{\partial x_i}\left[\rho \frac{\partial^2 \ln \rho}{\partial x_i \partial x_k}\right]. \tag{29}$$

In order to bring the RHS of equation (29) into a recognisable form let us write $\rho = R^2$. Then it is straightforward to show

$$\frac{1}{4m}\sum_i \frac{\partial}{\partial x_i}\left[\rho \frac{\partial^2 \ln \rho}{\partial x_i \partial x_k}\right] = \frac{1}{2m}\rho \frac{\partial}{\partial x_k}\sum_i \left[\frac{\partial^2 R}{\partial x_i^2}/R\right] \tag{30}$$

so that equation (28) becomes

$$\rho \frac{\partial}{\partial x_k}\left[\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + V - \frac{\hbar^2}{2m}\nabla^2 R/R\right] = 0. \tag{31}$$

Thus we arrive at the connection between the Moyal approach and the Bohm formalism as expressed through equation (17). Equation (31) is essentially equation (A 4.4) in Moyal's 1949 paper, the difference being that the equation (A 4.4) is derived for a charged particle moving in an electromagnetic field using the more general Hamiltonian $H(p_i, x_i) = \frac{1}{2m}\sum_i (p_i - eA_i)^2 + V(x_i)$. This reduces to equation (31) in the absence of the vector potential.

Hence the two essential equations that form the basis of the Bohm interpretation already appear in the appendix of Moyal's classic paper; Bohm providing a description of the individual, while the Moyal approach provides a description of the collective.

In passing it should be noted that by combining equations (27) and (28) we have

$$(\rho\, \overline{\overline{p_i p_k}}) - (\rho\, \overline{\overline{p_i}}.\, \overline{\overline{p_k}}) = -\frac{\hbar^2}{2}\frac{\partial}{\partial x_k}\left[\frac{\partial R}{\partial x_i}/R\right]. \tag{32}$$

Thus we see that, mathematically, the quantum potential arises as a consequence of the difference between the mean of the square of the momentum and the mean momentum squared. All this implies that the dispersion in the momentum for a single particle in quantum mechanics will, in general, be nonzero. For the single particle in classical physics the momentum is always dispersion free. In this way we see that the $\star$-product contains the structure that guarantees the existence of the uncertainty principle, contrary to what Dirac claims.

The connection with the quantum potential is made even clearer when one realises that the LHS of equation (28) is the total derivative of the mean momentum $\overline{\overline{p}}$, so that using equation (30) in equation (28) we find

$$m\frac{d\overline{\overline{v}}}{dt} = -\nabla[V + \hbar^2 Q_\psi] \tag{33}$$

where the quantum potential $\hbar^2 Q_\psi = -\frac{\hbar^2}{2m}\nabla^2 R/R$. Equation (33) explains the origin of the name since a force is derived from the quantum potential. Notice, once again, that $d\overline{\overline{v}}$ depends on the state ψ.

6 Conclusion

We have given a detailed account of the background to the problem of developing a theory involving a non-commutative phase space structure, showing exactly how the Moyal approach, generalised to include the Jordan product, fits into this theory. Because Moyal was concentrating on the statistical aspects he was led to the conservation of probability in the form of the quantum Liouville equation. Consequently he only produced half of the $\bigstar$-product necessary for a complete description of quantum phenomena.

On the other hand, Dirac was concentrating on the dynamical aspects of quantum phenomena and then seeing how the statistics arose. At that stage it was not clear how to bring these two aspects together, or even if that were possible. If this analysis is correct then one can begin to see how disagreements could arise.

The implications of von Neumann's approach to establishing the uniqueness of the Schrödinger picture are crucial here. It is not generally realised that von Neumann had shown it is possible to reproduce the expectation value of all functions of observables such as $g(\hat{P}, \hat{X}, t)$ by the method that Moyal was proposing. That is, simply by replacing them with functions of c-numbers, $g(p, x, t)$, and using the "distribution" function $F(p, x, t)$.

Unfortunately, this function has properties that disqualify it from being regarded as a probability because it takes on negative values when quantum effects arise. Rather than try to accommodate negative values into a theory about probabilities (Bartlett 1945, Feynman 1987), we need to focus on the meaning of the term "distribution" function when applied to $F(p, x, t)$.

In Dirac's (1945) own approach to developing a theory of the non-commutative phase space, he ended up with a distribution that was even worse because it gave complex probabilities. As Dirac remarked in his 1945 paper, at least the Moyal function was real but not positive. However, we can link the distribution $F(p, x, t)$ with the quantum density matrix through the relation

$$\rho(x, t) = \int F(x, p, t)\, dp = \psi(x, t)\, \psi^*(x, t) \tag{34}$$

and its generalisation. $\rho(x, t)$ is then identified as the density matrix of a pure state. This will always give a positive value and so can be used legitimately as a probability.

The reason why equation (34) works was pointed out by Baker (1958), who introduced the Jordan product in the form of what we have called the Baker bracket. If we form the Baker bracket with $\mathbf{a} = \mathbf{b}$ then the bracket simply becomes

$$\{\mathbf{a}, \mathbf{b}\}_{BB} = \mathbf{a} \bigstar \mathbf{a} = \mathbf{a}^2$$

so that an idempotent element produces the result $\{\mathbf{a}, \mathbf{a}\}_{BB} = \mathbf{a}$. Baker shows that the kernel, or propagator, is degenerate in the pure state, i.e. $K(x, y) = g^*(x)g(y)$. Recall the Baker bracket is simply the part of the $\bigstar$-product that Moyal does not consider in his discussion. My own research shows that the cosine bracket (13) leads to the conservation of energy. So ignoring this term will lead to nonphysical results.

In his attempt to show that Moyal's approach will not work, Dirac considers the problem that arises when converting a classical polynomial function of (p, x, t) into an operator; this is the well-known ordering problem. It is a difficulty that occurs in all approaches to quantum theory (see de Gosson (2016)). It arises because, in the Schrödinger picture, we have the replacement $\hat{X} \to x$, but $\hat{P}_x \to -i\hbar\partial/\partial x$, so clearly $\hat{X}\hat{P}$ will differ from $\hat{P}\hat{X}$ by a factor $\hbar$.

However, this can be taken care of by choosing the appropriate standard ket introduced by Dirac himself. This special ket plays the role of a vacuum state, so by choosing the vacuum state correctly, we find the ordering problem disappears. For example the normal ordering chosen in the Heisenberg picture starts with no energy in the vacuum state, whereas in the Weyl ordering, the energy in the vacuum state is $\hbar\nu/2$. Thus the zero-point energy is correctly specified by the Weyl ordering (Hirshfeld and Henselder 2002b).

In one final attempt to show that the Moyal approach will not work, Dirac (in Dirac and Moyal 1944–1946, p.147) writes: "Your theory gives correctly the average energy when the system is in a given state (i.e. represented by a given wave function) but not when the system is at a given temperature." He then proceeds in the letter to show that it gives $\bar{E} = kT$ which is, unfortunately, the wrong result. It should be

$$\bar{E} = \hbar\nu(e^{\hbar\nu/kT} - 1)^{-1} + \hbar\nu/2.$$

The full treatment of this problem appears in Bartlett and Moyal (1949).

Unfortunately, in his reply to Dirac's letter, Moyal refers to an early draft of the Bartlett and Moyal paper of which I have no copy. Dirac's response (Dirac and Moyal 1944–1946, p.147) to the draft of that paper is that "the quantum values for the energy of the harmonic oscillator are *assumed* and the correct value for $\bar{E}$ was obtained because of this assumption." Here Dirac seems to have misunderstood the argument proposed by Bartlett and Moyal (1949) and all the correct results emerge directly from the theory proposed by Moyal. An independent and simple verification of this result can be found in Case (2008). Unfortunately, Dirac got this one wrong, as shown in the comprehensive work of Curtright, Fairlie and Zachos (2014).

I do not want to give the impression that I am trying to identify the hero and the villain in this controversy. Both Moyal and Dirac were working outside the box of "orthodoxy," grappling with the deep implications of a new non-commutative dynamics. Both were pioneers and I have certainly learned a lot by studying their disagreements. Dirac is, of course, a giant in physics and his work has been indispensable in the development of quantum theory. I have a great empathy for Moyal, given the turmoil of his life in the nineteen-forties, a turmoil I experienced making one of the last crossings of France while travelling to meet up with my father.

While Dirac's work was taken up by Feynman and developed into a very successful quantum electrodynamics, the importance of Moyal's work in laying the foundations of the $\star$-algebras has been slowly gathering pace (Curtright, Fairlie and Zachos 2014). The growing interest has coincided with the ongoing development of non-commutative geometries as demonstrated in the books of Connes (1994) and Madore (1995). One particular use of Moyal's work in non-commutative quantum field theory has been discussed in Gayral et al. (2004).

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